Exam Elementary Particles
June 21, 2023
Start: 15:00h End: 17:00h

Each sheet with your name and student ID

INSTRUCTIONS: This is a closed-book and closed-notes exam. You are allowed to bring one A4 page written by you on both sides, with useful formulas. The exam duration is 2 hours. There is a total of 9 points that you can collect. Use the official exam paper for *all* your work and ask for more if you need.

1. **(3 points)** Consider the $SU(N)$ Yang-Mills theory coupled to Dirac fermions in the fundamental representation of $SU(N)$. Under the local gauge transformation and employing the vector notation, one has:

$$\psi(x)' = U(x)\psi(x) \tag{1}$$

and

$$D_\mu\psi(x)' = U(x)D_\mu\psi(x) \tag{2}$$

with $D_\mu\psi = (\partial_\mu + A_\mu)\psi$ and $U(x)$ a special unitary matrix. Show that the transformation of the gauge field $A_\mu(x)$ implied by (2) under $SU(N)$ local is:

$$A_\mu(x)' = U(x)A_\mu(x)U^\dagger(x) - (\partial_\mu U(x))U^\dagger(x) \tag{3}$$

2. **(3 points)** The Lagrangian density of the Fermi theory for leptonic weak interactions reads:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \tag{4}$$

where

$$\mathcal{L}_0 = \sum_{l=e,\mu,\tau} \bar{\psi}^{(\nu_l)}(x) i\partial\!\!\!/ \psi^{(\nu_l)} + \sum_{l=e,\mu,\tau} \bar{\psi}^{(l)}(x)(i\partial\!\!\!/ - m_l)\psi^{(l)} \tag{5}$$

is the Dirac Lagrangian for free massless neutrinos and massive charged leptons, and:

$$\mathcal{L}_I = \frac{G_F}{\sqrt{2}} J^\mu J^\dagger_\mu \tag{6}$$

is the 4-fermion interaction Lagrangian with Fermi coupling G_F and the weak leptonic current:

$$J_\mu = \sum_{l=e,\mu,\tau} \bar{\psi}^{(\nu_l)}(x)\gamma_\mu(1-\gamma_5)\psi^{(l)}(x) \tag{7}$$

a) **(1.5 points)** Derive the canonical dimensions of fields and couplings in generic d spacetime dimensions. [**Show your work**]

b) **(1.5 points)** For which spacetime dimension is the theory renormalizable? [**Show your work**] i.e. explain your reasoning by also employing the superficial degree of divergence expression.

→ **Bonus Question (1 point)** For the renormalizable theory in **b)**, find the finite set of superficially divergent amplitudes, i.e. draw these amplitudes and determine the corresponding value for D.

Useful formulas:

- The superficial degree of divergence formula for this theory in d spacetime dimensions reads:

$$D = d - [coupling]V - [\psi]E_\psi \tag{8}$$

where $[\cdots]$ stands for the dimension of its argument, V is the number of vertices and E_ψ is the number of external fermionic lines.

3. (3 points total) The flavour-changing process $b \to s\gamma$ does not occur at tree level in the Standard Model. It occurs at one loop and one of the dominant contributions is a loop involving the top quark and the W boson as depicted in Figure 1.

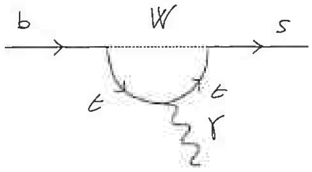

Figure 1: A one-loop Standard Model contribution to $b \to s\gamma$.

The relevant interaction Lagrangian reads:

$$\mathcal{L}_I = \frac{e}{\sqrt{2}\sin\theta_W}(W^+_\mu J^\mu_+ + W^-_\mu J^\mu_-) + eA_\mu J^\mu_{em} \tag{9}$$

with currents

$$\begin{aligned} J^\mu_+ &= V_{ij}\bar{u}^i_L\gamma^\mu d^j_L = \frac{V_{ij}}{2}\bar{u}^i\gamma^\mu(1-\gamma_5)d^j \\ J^\mu_- &= (J^\mu_+)^\dagger = (V^\dagger)_{ij}\bar{d}^i_L\gamma^\mu u^j_L = \frac{(V^\dagger)_{ij}}{2}\bar{d}^i\gamma^\mu(1-\gamma_5)u^j \\ J^\mu_{em} &= \sum_i Q_i\bar{q}_i\gamma^\mu q_i \end{aligned} \tag{10}$$

where quarks are massive and currents as well as Figure 1 are written in terms of the up- and down-quark <u>mass</u> eigenstates:

$$\begin{aligned} u^i &= \{u, c, t\}, \quad i = 1, 2, 3 \\ d^i &= \{d, s, b\}, \quad i = 1, 2, 3 \end{aligned} \tag{11}$$

so that V_{ij} is the corresponding (ij) entry of the CKM mixing matrix V for the three quark generations.

a) (1 point) Write the relevant Feynman rules in momentum space in terms of Dirac spinors by employing the Lagrangian density (9).

b) (2 points) Write the amplitude for the diagram in Figure 1.

Hints:

- Given the Dirac spinor q, one has: $q = q_L + q_R$, with $q_{L,R} = P_{L,R}q$ and the projectors $P_L = (1-\gamma_5)/2$ and $P_R = (1+\gamma_5)/2$.
- Electric charges are $Q_i = 2/3$ for up quarks and $Q_i = -1/3$ for down quarks in units of $e \equiv |e|$.

- There is an integral $\int \frac{d^4k}{(2\pi)^4}$ for each independent internal momentum k.
- Employ massive propagators for the quarks and:

$$iD_{\mu\nu}(k, M_W) = i\frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_W^2}}{k^2 - M_W^2 + i\varepsilon} \tag{12}$$

  for the massive $W^\pm$ boson.